Chaology of quantum maps applied to the Standard map

Edson M. Signor[1]
[1]*Department of Physics, University of Connecticut, Storrs, CT, USA*
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I. INTRODUCTION & ABSTRACT

This tutorial concerns to give a basic introduction to the concept of classical and especially quantum maps. Due to their simplicity and easy manipulation, quantum maps have been a great resource to investigate some intrinsic questions in quantum chaos. Here, we present the map quantization in the torus and apply it in spectral analysis as spacing distribution. Furthermore, we bring the Husimi distributions to illustrate the "quantum" phase space of the quantum maps eigenstates. The calculations are applied to the quantum standard map but can be easily employed on any other map.

II. CLASSICAL MAPS

Studying the dynamics of the Hamiltonian system can be rather complicated, especially when it evolves in high dimensions. However, maps provide a more accessible approach since they only regard successive iterations. Nowadays, there are plenty of different maps for specific research; here, we focus on the ones constructed from kicked Hamiltonians.

The general equation for kicked systems is

$$H(q,p) = f(p) + \sum_{n=-\infty}^{\infty} \delta(t/\tau - n)V(q) \tag{1}$$

where $f(p)$ is the free kinetic term, τ is the period that each kick takes place and $V(q)$ is the potential produced by the kick. Usually, in $V(q)$, we have a parameter that controls the kick intensity, which means it controls the perturbation. For instance, the well-known kicked rotor has

$$f(p) = \frac{p^2}{2}, \quad V(q) = -\frac{g}{4\pi^2}\cos(2\pi q) \tag{2}$$

where g is the kick-intensity parameter.

To obtain the map from the Hamiltonian, we just need to apply Hamilton equations and integrate them over a period

$$\int_{n^+}^{(n+1)^+} \frac{dp}{dt}dt = \int_{n^+}^{(n+1)^+} -\frac{\partial H}{\partial q}dt \qquad \int_{n^+}^{(n+1)^+} \frac{dq}{dt}dt = \int_{n^+}^{(n+1)^+} \frac{\partial H}{\partial p}dt \tag{3}$$

the "+"("-") means that the time is right after (before) the kick. We can split the time into two: the free dynamics $n^+ < t/\tau < (n+1)^-$ and the impulsive part in $(n+1)^- < t/\tau < (n+1)^+$. We have the 4 equations

$$\int_{n^+tau}^{(n+1)^-} \frac{dp}{dt}dt = 0, \qquad \int_{n^+}^{(n+1)^-} \frac{dq}{dt}dt = \int_{n^+}^{(n+1)^+} \frac{df(p)}{dp}dt = \tau f'(p_n), \tag{4}$$

$$\int_{(n+1)^-}^{(n+1)^+} \frac{dp}{dt}dt = -\int_{(n+1)^-}^{(n+1)^+} \sum_{n=-\infty}^{\infty} \delta(t/\tau - n)V(q)dt = -\tau V'(q_{n+1}^+), \qquad \int_{(n+1)^-}^{(n+1)^+} \frac{dq}{dt}dt = 0. \tag{5}$$

Finally, we combine the solutions and obtain the general iterating map for kicked systems

$$q_{n+1} = q_n + \tau\frac{d}{dp}f(p_n) \tag{6}$$

$$p_{n+1} = p_n - \tau\frac{d}{dp}V(q_{n+1}). \tag{7}$$

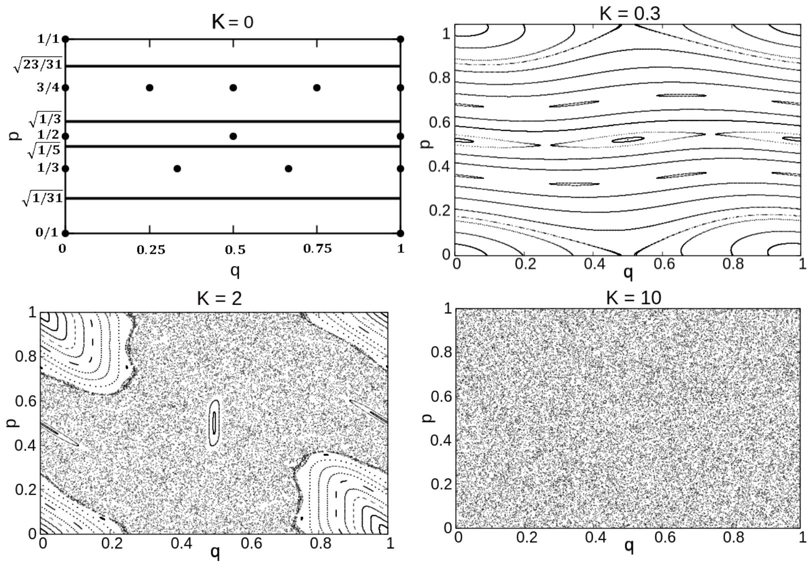

FIG. 1. Phase space of the standard map for different values of K.

If we consider the kicked rotor from 2, we have the well-known standard map

$$q_{n+1} = q_n + p_n \qquad \mod 1 \tag{8}$$

$$p_{n+1} = p_n - \frac{K}{2\pi}\sin\left(2\pi q_{n+1}\right) \qquad \mod 1 \tag{9}$$

where $K = \tau^2 g$ and $\tau p \to p$. The modulus, in the generalized position, is regarding the angle position of the rotor, but we restrict the momentum because of the system symmetry (Exercise 2). Therefore, the phase space is a two-dimensional torus.

Exercises:

1. Calculate the Jacobian of the Standard map and show that it is an area-preserving map, i.e., $det(J) = 1$.

$$J = \begin{pmatrix} \frac{\partial q_{n+1}}{\partial q_n} & \frac{\partial q_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial q_n} & \frac{\partial p_{n+1}}{\partial p_n} \end{pmatrix} \tag{10}$$

2. Momentum symmetry. See that the Standard map is invariant about a momentum translation by one step ($p \to p + 1, q \to q$).

With the standard map, it becomes easy to iterate in a computer and study the phase space for many different initial conditions. In figure 1, we have four samples of the phase space for different values of perturbation. Along with the maps, one can study the perturbation on KAM tori, seeing $K = 0 \to K = 0.3$, bifurcations, and mixed phase space for $K = 2$, or when the system is chaotic $K = 10$.

Exercise:

1. Create a code iterating the standard map for $K = 0.3, 2$, and 10. For each parameter, make ~ 200 different initial conditions and iterate them ~ 100 times. Plot the phase space and see the richness of structures with a simple code. (Remember to take the modulus of q and p after each iteration).
   Hint: take a look at the code ***Std_map.py***

III. QUANTUM MAPS

Now, we approach the quantum equivalent of the classical maps. To illustrate, we basically need an operator that takes a wavefunction from the state $t/\tau = n$ to $t/\tau = n+1$. As it is well-known, the time translator in quantum mechanics is the evolution operator

$$U(t) = e^{-iHt/\hbar}. \tag{11}$$

Since we are interested only after each period, we have

$$|\psi((n+1)^+)\rangle = U|\psi(n^+)\rangle. \tag{12}$$

Taking the Hamiltonian of kicked system 1, we can determine U by applying the Hamiltonian into Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{13}$$

To solve it, we have to split the time between kicks in the same way we did for the classical case. Then,

$$[n^+,(n+1)^-]: \quad i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = f(p)|\psi(t)\rangle \tag{14}$$

$$[(n+1)^-,(n+1)^+]: \quad i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \sum_{n=-\infty}^{\infty}\delta(t/\tau - n)V(q)|\psi(t)\rangle \tag{15}$$

Integrating both equations in their corresponding intervals and combining them, we got

$$|\psi((n+1))\rangle = \exp\left(-iV(q)\tau/\hbar\right)\exp\left(-if(p)\tau/\hbar\right)|\psi(n)\rangle \tag{16}$$

which is called the quantum map. Therefore, substituting 2, the quantum standard map becomes

$$U = \exp\left(\frac{iK}{4\pi^2\tau\hbar}\cos\left(2\pi q\right)\right)\exp\left(\frac{ip^2\tau}{2\hbar}\right) \tag{17}$$

From the kinetic part of the Hamiltonian, we can see that the rotor has a moment of internal equals to one $I = 1$. In general, we should include it and rescale the $\hbar$ properly in order to be dimensionless. So, a semiclassical limit $\hbar \to 0$ can be taken to study and compare with classical results. Since I/τ has action dimension, we define

$$\hbar^* = \frac{\hbar}{(I/\tau)}. \tag{18}$$

In our case, $I = 1$, and we set $\tau = 1$ to be even simpler.

However, in the case of the quantum standard map, we are restricted to a torus. The momentum and position are confined $0 \leq q, p < 1$, which leads to a finite number in quantization. The Hilbert space is not infinite anymore. Besides that, due to the periodicity, we have the following condition for the wavefunction

$$|\psi(q+1)| = |\psi(q)|, \quad |\psi(p+1)| = |\psi(p)| \tag{19}$$

or

$$\psi(q+1) = e^{2\pi i\beta}\psi(q), \quad \psi(p+1) = e^{-2\pi i\alpha}\psi(p), \qquad 0 \leq \alpha, \beta < 1. \tag{20}$$

The additional phases brought from the torus topology are known as quantum phases and lack classical analog. They are purely quantum and control the symmetry of the system. For $\alpha = 0$, the parity symmetry is preserved; otherwise, it is broken. Similarly, for $\beta = 0$, the time-reversal symmetry is preserved, and it is broken for any other value in the interval.

Getting back to position and momentum basis, we need to include those phases gained in each cycle. We say that the finite number of q and p is N, then the possible values are

$$\begin{aligned} \{q\}: &\quad 0/N, 1/N, \ldots n/N, \ldots, (N-1)/N \\ \{p\}: &\quad 0/N, 1/N, \ldots n/N, \ldots, (N-1)/N \end{aligned}$$

Including the phases, we can set the basis of the q and p

$$\begin{aligned} \{|q_n\rangle\}: &\quad |(0+\alpha)/N\rangle, |(1+\alpha)/N\rangle, \ldots |(n+\alpha)/N\rangle, \ldots, |((N-1)+\alpha)/N\rangle \\ \{|p_n\rangle\}: &\quad |(0+\beta)/N\rangle, |(1+\beta)/N\rangle, \ldots |(n+\beta)/N\rangle, \ldots, |((N-1)+\beta)/N\rangle \end{aligned}$$

where

$$q|q_n\rangle = \frac{n+\alpha}{N}|q_n\rangle \tag{21}$$

$$p|p_m\rangle = \frac{m+\alpha}{N}|p_m\rangle. \tag{22}$$

Finally, to interchange between the bases, we have the following Fourier relation

$$\langle q_n|p_m\rangle = \frac{1}{N}\exp\left(\frac{2\pi i}{N}(n+\alpha)(m+\beta)\right). \tag{23}$$

If you are interested in proving all these relations, I recommend Lakshminarayan's notes [1] from page 54 to 56. The finitude of the Hilbert space has another considerable impact. Since each quantum state occupies a volume of h or $2\pi\hbar$ and we just learned that there are only N positions and momenta, the ratio of the total area of phase space (which is unitary) and the number of states (which is N) is equal to $2\pi\hbar$. Then,

$$2\pi\hbar = 1/N. \tag{24}$$

The Semiclassical limit is when the Hilbert space dimension tends to infinity $N \to \infty$.

The dynamics of the quantum system are contained in the quantum map 17, since we just need to act the operator as many times we want to evolve an initial state

$$|\psi(n)\rangle = U^n|\psi(0)\rangle. \tag{25}$$

When the problem relates to the power of operators, essentially, we just solve an eigenvalue problem. Introducing the eigenvalues and eigenstates of U

$$U|\phi_k\rangle = e^{i\theta_k}|\phi_k\rangle, \quad k = 0, \ldots, N-1 \tag{26}$$

we can easily solve the dynamics

$$|\psi(n)\rangle = \sum_{k=0}^{N-1} e^{i\theta_k n}\langle\phi_k|\psi(0)\rangle\, |\phi_k\rangle. \tag{27}$$

The θ_k are called eigenangles or quasienergies since they perform a similar role as the energies in the usual sense. They receive this name because the Hamiltonian is time-dependent, and the quasienergies are stationary in a stroboscopic way. Likewise, the eigenstates $|\phi_k\rangle$ are also stationary in the period and are called Floquet states.

To determine the eigenvalues and eigenstates, we need the operator U in a certain base. We will show the U in position representation, but a similar calculation can be made in momentum representation. So, we have

$$U_{n'n} = \langle q_{n'}|U|q_n\rangle = \langle q_{n'}|\exp\left(\frac{iK}{4\pi^2\tau\hbar}\cos(2\pi q)\right)\exp\left(\frac{ip^2}{2\hbar}\right)|q_n\rangle \tag{28}$$

$$= \exp\left(\frac{iK}{4\pi^2\hbar}\cos\left(\frac{2\pi}{N}(n^{'}+\alpha)\right)\right)\sum_{m=0}^{N-1}\langle q_{n'}|p_m\rangle\langle p_m|\exp\left(\frac{ip^2}{2\hbar}\right)|q_n\rangle \tag{29}$$

$$= \exp\left(\frac{iK}{4\pi^2\hbar}\cos\left(\frac{2\pi}{N}(n^{'}+\alpha)\right)\right)\sum_{m=0}^{N-1}\exp\left(\frac{i}{2\hbar N^2}(m+\beta)^2\right)\langle q_{n'}|p_m\rangle\langle p_m||q_n\rangle \tag{30}$$

$$= \frac{1}{N}\exp\left(\frac{iK}{4\pi^2\hbar}\cos\left(\frac{2\pi}{N}(n^{'}+\alpha)\right)\right)\sum_{m=0}^{N-1}\exp\left(\frac{i}{2\hbar N^2}(m+\beta)^2\right)\exp\left(\frac{2\pi i}{N}(m+\beta)(n^{'}-n)\right) \tag{31}$$

For periodic boundary conditions $\alpha = \beta = 0$, we can apply the following Gaussian sum

$$\sum_{m=0}^{N-1} \exp\left(-\frac{i\pi}{N}(m^2 - 2m(n^{'} - n))\right) = \sqrt{N}e^{i\pi/4}\exp\left(\frac{i\pi}{N}(n^{'} - n)^2\right) \tag{32}$$

and we have the simple matrix

$$U_{n^{'}n} = \frac{e^{i\pi/4}}{\sqrt{N}}\exp\left(\frac{iKN}{2\pi}\cos\left(\frac{2\pi(n^{'} + \alpha)}{N}\right)\right)\exp\frac{i\pi}{N}(n^{'} - n)^2. \tag{33}$$

Doing the diagonalization of the matrix 31, we have its eigenvalues, and a known measure of the level rigidity of chaotic systems is the level spacing distribution. We can easily change the value of the perturbation parameter K from near-integral, mixed, and chaotic in order to verify the conjectures of their spectra with Random matrix theory. The spacing is defined as

$$s_i = \theta_i - \theta_{i-1}, \quad i = 1, N-1 \tag{34}$$

but to compare different distributions, we need to have a unitary mean spacing $\bar{s}$. For the case of the quantum maps, since their eigenvalues are distributed around the unitary circle

$$\bar{s} = \frac{2\pi}{N}, \tag{35}$$

so

$$s_i = \frac{N}{2\pi}(\theta_i - \theta_{i-1}). \tag{36}$$

In figure 2, the distributions were constructed for the same parameters as 1. For small K, the distribution coincides with a Poisson distribution, as stated by the Berry-Tabor conjecture. The spectrum of near-integrable systems has uncorrelated eigenvalues. As we increase the perturbation parameter, the distribution tends to the circular unitary ensemble of Random matrices, as stated by the Bohigas-Gionanni-Smith conjecture. Despite the orthogonal ensemble, for these results, we have settled on a broken time-reversal symmetry in which the proper ensemble is the unitary with complex values. In summary, we can rely on the following equations

$$P(s) = \begin{cases} \exp(-s) & \text{Integrable regime} \\ \frac{\pi}{2}s\exp(\pi s^2/4) & \text{Chaotic regime with TR symmetry} \\ \frac{32}{\pi^2}s\exp(4s^2/\pi) & \text{Chaotic regime without TR symmetry} \end{cases} \tag{37}$$

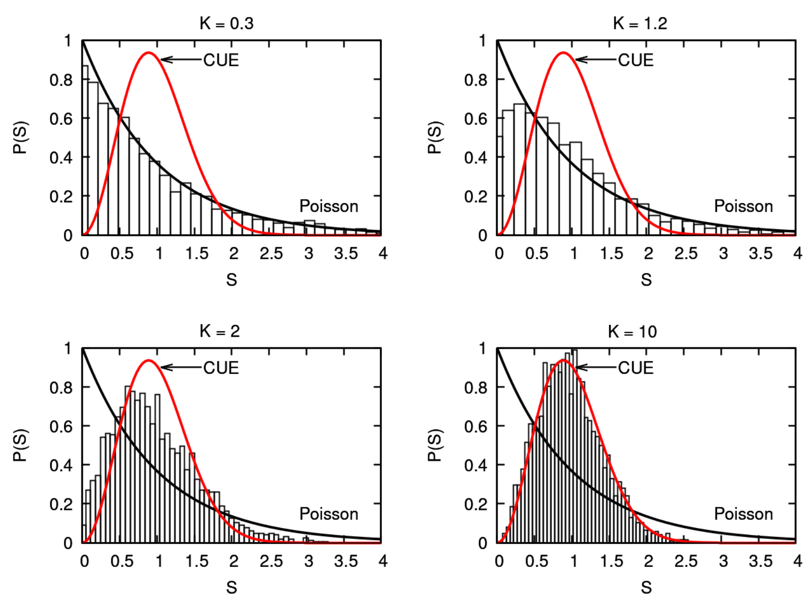

FIG. 2. Level spacing of the quantum standard map for different values of K (compare with figure 1). For this result, it was set $N = 2000, \alpha = \beta = 1/4$.

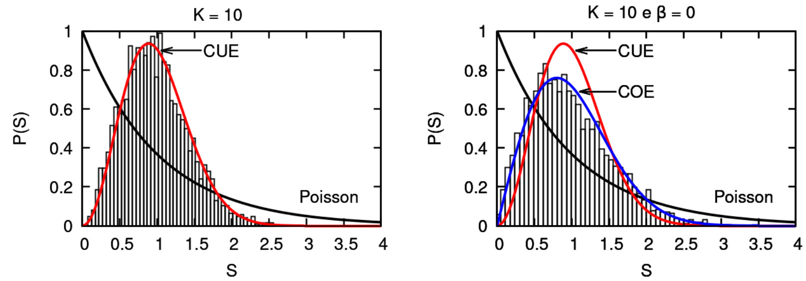

FIG. 3. Level spacing of the quantum standard map for the chaotic regime $K = 10$. Whereas the quantum phases are $\alpha = \beta = 1/4$ in the left panel, we preserved TR symmetry in the right panel: $\alpha = 1/4$ and $\beta = 0$. For these results, it was set $N = 2000$.

Finally, in the figure 3, we can easily see the spectrum dependence on the TR symmetry controlled by the quantum phase β. For broken symmetry $\beta = 1/4$, the distribution corresponds to the circular unitary ensemble. When the symmetry is conserved for $\beta = 0$, the distribution corresponds to the orthogonal ensemble.

Exercise:

1. From the Evolution operator in position representation of the quantum standard map, set the matrix in your preferred language code and calculate its eigenvalues. Check that all fall at the unitary circle. (Suggested parameters $K = 2, N = 1000, \alpha = \beta = 0$)
   Hint: take a look at the code ***Eigenvalues_complex_plan.py***

2. After problem 1, calculate each correspondent angle of the eigenvalues. Sort them and calculate the level spacing (Make sure to divide by $N/2\pi$). After that, reproduce the distributions for the same values of K, α, and β (you can use $N = 1000$ or 2000). For $K = 10$, vary just α and see that the level spacing distribution does not change. Hint: take a look at the code ***level_spacing_distribuition.py***

3. Now, calculate the eigenvectors and plot $|\psi|^2$ vs q for $K = 0.1, 2$, and 10. The states tell the classical story of the standard map. (Use $N = 1000, \alpha = \beta = 0$.) This becomes clearer when the Husimi distributions are applied. Hint: take a look at the code *Eigenvectors_QSM_quantum.py*. Just apply $||^2$ and plot the result.

Besides Lakshminarayan's notes [1] being a very good resource for both classical and quantum maps, I also recommend Bäcker proceeding [2], which covers the same topic in a very understandable way.

A. Bonus: Husimi distribution

In this section, we extend the discussion of quantum maps to briefly show how to calculate the Husimi distribution of states that are constrained in a toroidal space. First, we need the coherent state $|z\rangle$ definitions since the Husimi is

$$H(q,p) = \frac{1}{\langle z|z\rangle}|\langle z|\psi\rangle|^2 \tag{38}$$

where

$$|z\rangle = e^{\bar{z}a^\dagger|0\rangle}, \quad |0\rangle \text{ is GS of Harmonic Oscillator,} \tag{39}$$

$$z = \frac{1}{\sqrt{2}}(q - ip), \tag{40}$$

and the creation operator is

$$a^\dagger = \frac{1}{\sqrt{2}\hbar}(q - ip). \tag{41}$$

Finally, we will need the ket $|z\rangle$ in the position basis

$$\langle q|z\rangle = \frac{1}{\pi\hbar}\exp\left\{\frac{1}{\hbar}\left[-\frac{\bar{z}^2+q^2}{2} + \sqrt{2}\bar{z}q\right]\right\}. \tag{42}$$

In order to calculate $H(q,p)$, we introduce the position complete set

$$H(q,p) = e^{-2\pi N|z|^2}\left|\sum_{j=0}^{N-1}\langle z|q\rangle_T\, {}_T\langle q_j|\psi\rangle\right|^2. \tag{43}$$

However, we need to take into account the periodicity of the position in the torus space. So,

$$\langle z|q_j\rangle_T = \sum_{\nu=-\infty}^{\infty} e^{2\pi i\nu\beta}\langle z|q_j+\nu\rangle. \tag{44}$$

Therefore, we just substitute 42 into the toroidal bracket and calculate the Husimi

$$H(q,p) = \sqrt{2N}\left|\sum_{j=0}^{N-1}\exp\left(-\pi N(q^2 - ipq)\right)\exp\left(\pi N(-q_j^2 + 2(q-ip)q_j)\right)\theta_3\left(i\pi N\left(q_j - \frac{i\beta}{N} - q + ip\right)\middle| iN\right)\right|^2 \tag{45}$$

where θ_3 is the Jacobi theta function defined as

$$\theta_3(\xi|\tau) = \sum_{\nu=-\infty}^{\infty} e^{i\pi\tau\nu^2 + 2i\nu\xi}. \tag{46}$$

In figure 4, we plotted some examples of Husimis of the quantum standard map. The inset shows the respective intensity plots $|\psi(q)|^2$. In the near-integrable regime with $K = 0.1$, the eigenstates are trapped by the KAM tori (upper panel), and some already formed stable islands (lower panel). The $K = 2.05$ results are divided by chaotic and island states, although a mixed state seldom occurs. The fully chaotic parameter $K = 10$ has states distributed over all quantum phase space. The comparison with the classical map in figure 1 is enticing.

For further information about the Husimi distribution of quantum maps, I recommend the [3], which covers toroidal, cylindrical, and spherical quantization and Husimi distributions. In addition, there are two codes in python ***level_spacing_distribuition.py*** and ***Husimi_QSM.py*** where you can reproduce the Husimis. The former is to create the eigenvectors, and the latter is to construct the Husimi of the chosen eigenstate.

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[1] A. Lakshminarayan, Classical and quantum chaos plus rmt and some applications (ICTS, Bangalore, 2018).

[2] A. Bäcker, Numerical aspects of eigenvalue and eigenfunction computations for chaotic quantum systems, in *The Mathematical Aspects of Quantum Maps*, edited by M. D. Esposti and S. Graffi (Springer Berlin Heidelberg, Berlin, Heidelberg, 2003) pp. 91–144.

[3] V. A. Leboeuf, P., Quantum nodal points as fingerprints of classical chaos, IPNO-TH **19**, 92 (1992).

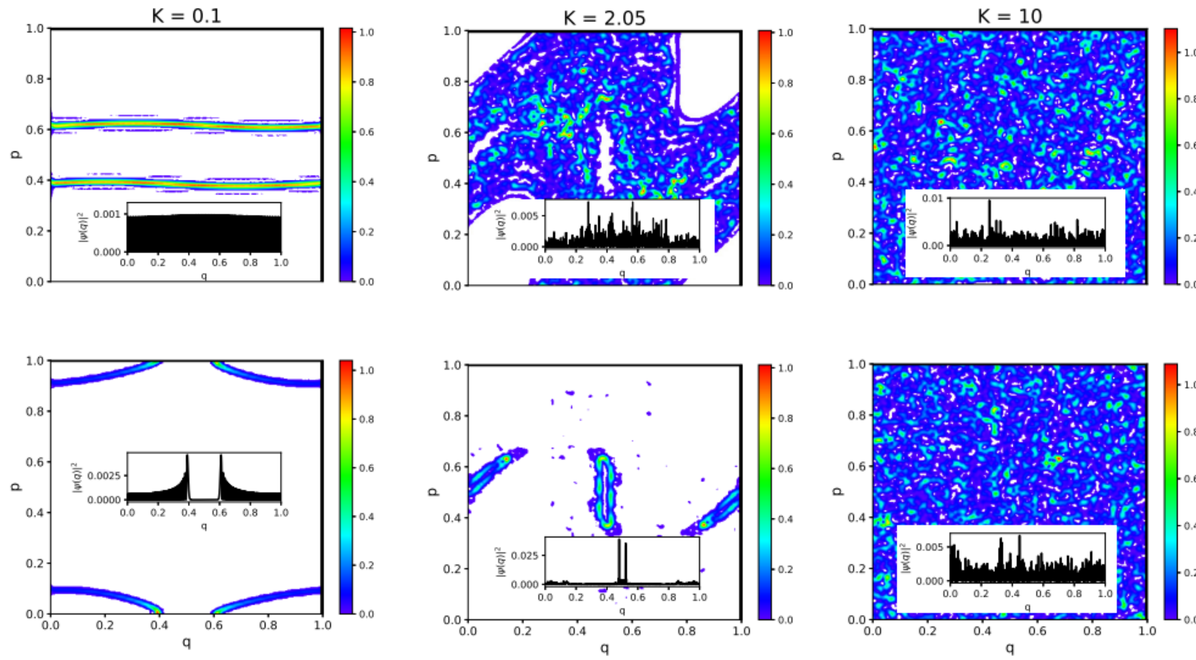

FIG. 4. Husimi distribution of typical states with $N = 2098$, $\alpha = 1/4$, and $\beta = 0$.