Leaking Classical and Quantum Systems: Applied to the Standard map

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Motivation

There are a numerous physical situation in which a HOLE or LEAK is introduced in an otherwise closed system.

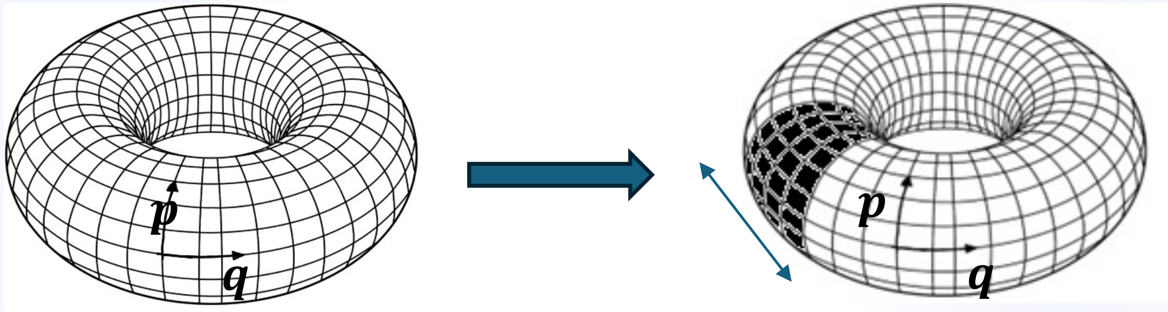
- Natural origin.
- It can mimic measurement devices.
- It can also be used to reveal dynamical properties of closed system

Applications: Room acoustic, Billiards, Chemical reactions, Hydrodynamical flow, Planetary science, Optical microcavities, Plasmas physics

DISSIPATION

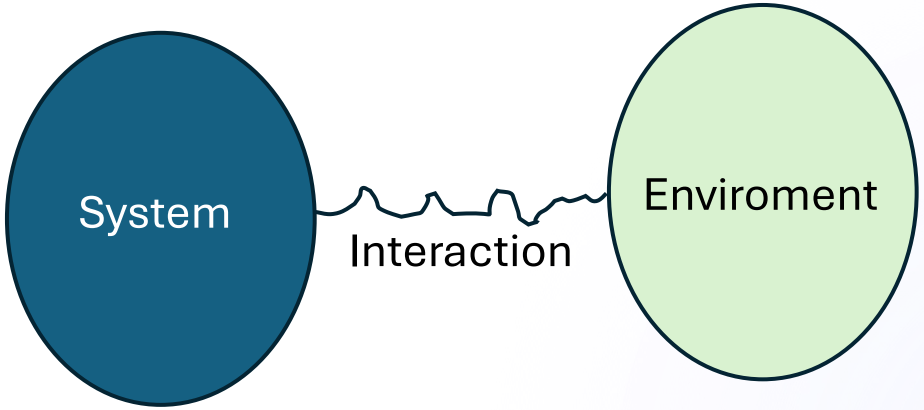
LEAKING SYSTEM

- Escape or removal of trajectories
- Conservative system remain conservative after becoming leaky.
- Persistent chaos becomes transient chaos.



INTERACTION WITH ENVIROMENTS

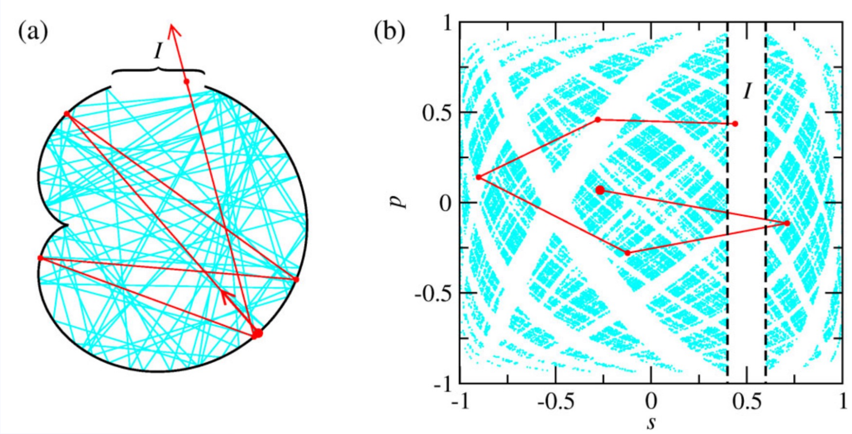
- Exchange of energy, matter or information with the environment
- Contraction in the phase space
- The dynamics of the environment can influence the system



Classical Methods

LEAKING SYSTEM

- Analysis of trajectories in phase space

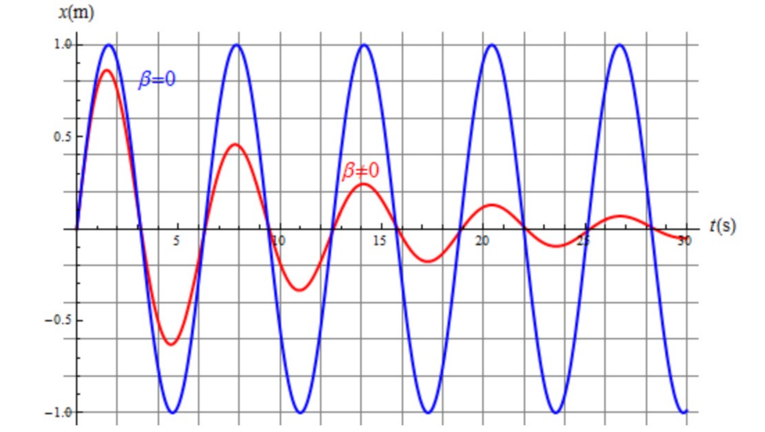


- Sabine’s law

INTERACTION WITH ENVIROMENTS

- Effect of environment using differential equations

$$x''(t) + \beta x'(t) + \omega^2 x(t) = 0,$$



- Fokker-Planck equation

Quantum Methods

LEAKING SYSTEM

- Eigenvalue problem for effective evolution operator

(Non-Hermitian Problem)

Second part for Edson

INTERACTION WITH ENVIROMENTS

$$H = H_S + H_E + \mu I$$

System

enviroment

Interaction:

- Master equation for the density matrix:

Lindblad Master equation (Devesh's presentation):

$$\frac{d\rho}{dt} = -i[H, \rho] + \sum_i \gamma_i \left[L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\} \right]$$

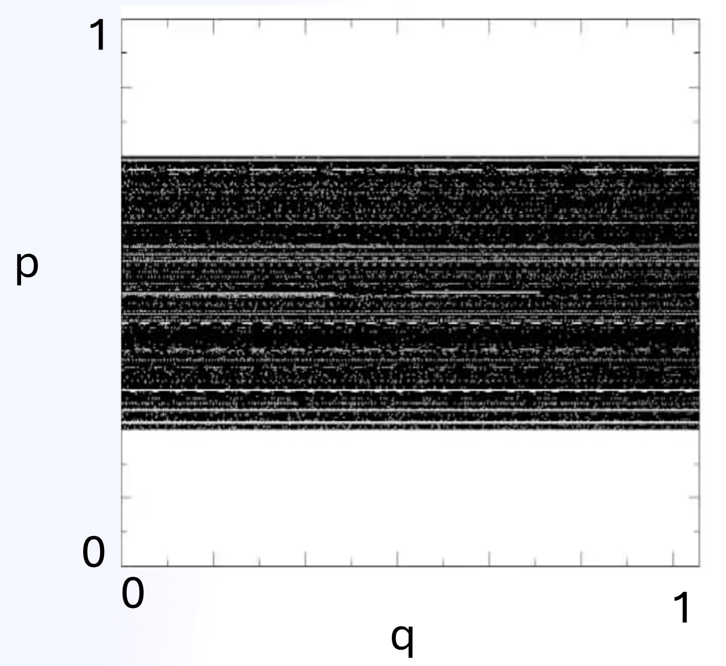
- Non-Hermitian Hamiltonian with a complex term

Our system: Standard Map in the Chaotic Regime

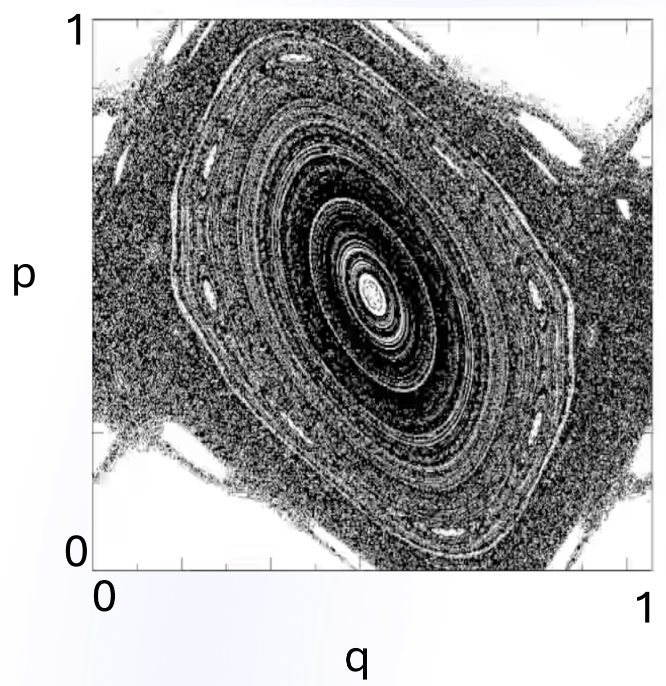
Classical map

$$q_{n+1} = q_n + p_n \qquad \pmod 1$$
$$p_{n+1} = p_n - \frac{K}{2\pi}\sin(2\pi q_{n+1}) \qquad \pmod 1$$

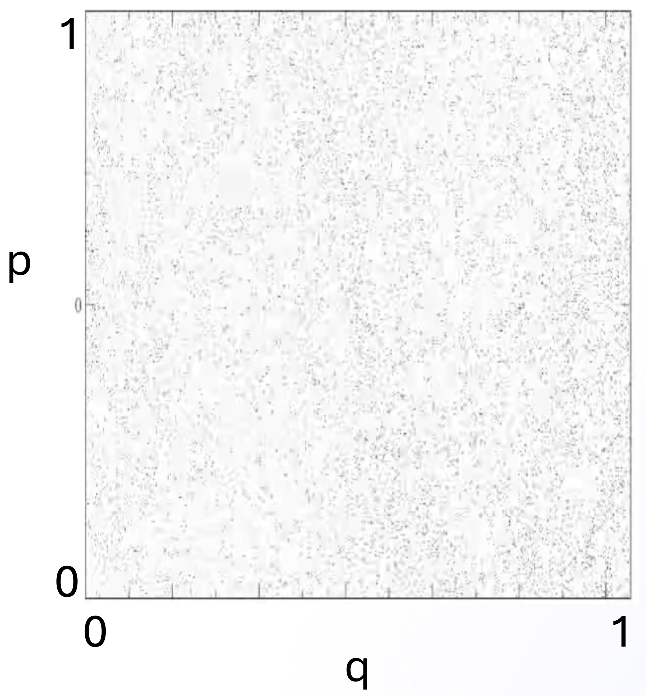
K=0



K=1

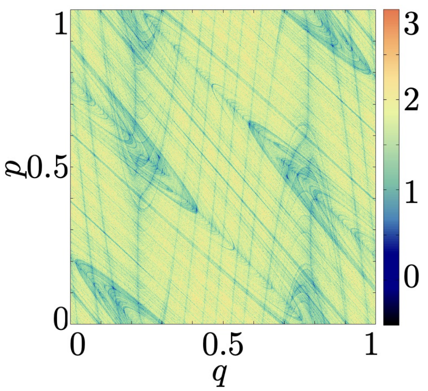


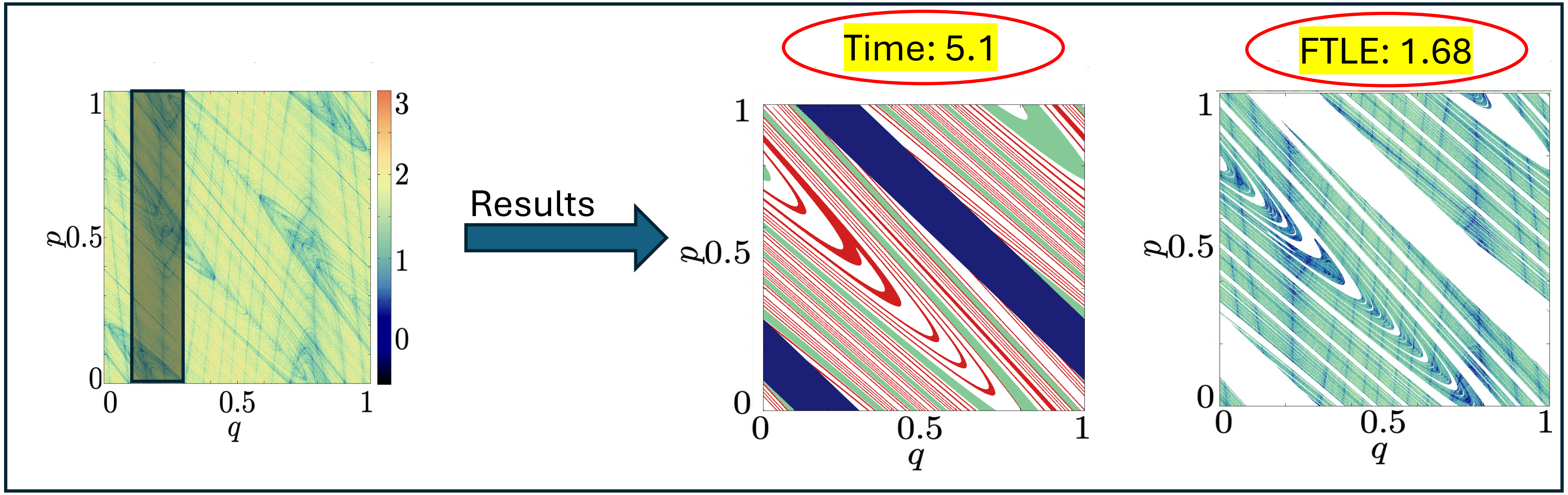
K=10

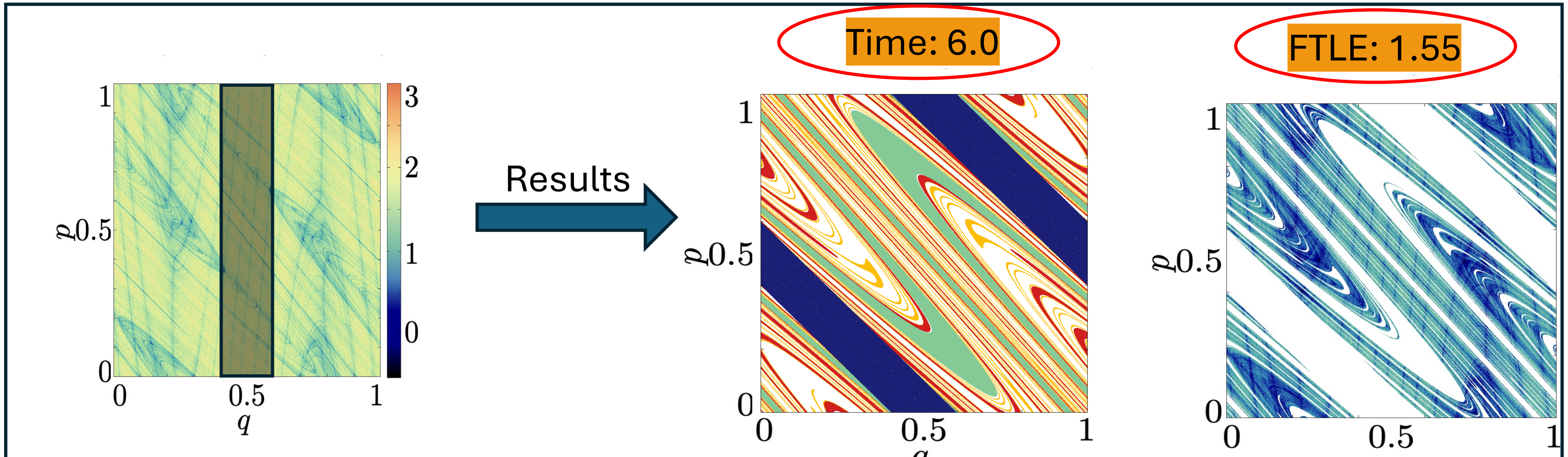


Strong Chaos

Leaky Standard Map in the Chaotic Regime

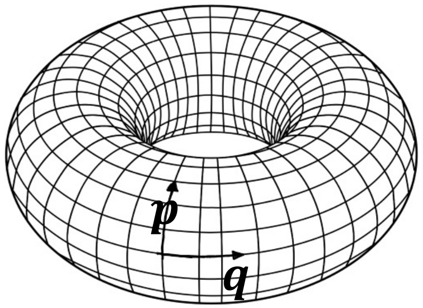






Review quantum maps

- Quantum map U: $|\psi(n+1)\rangle = U\,|\psi(n)\rangle, \qquad \tau = 1$



- Position and momentum basis: $\alpha = \beta = 0$ (periodic conditions)

$$\{|q_n\rangle\} = |0\rangle, |1/N\rangle, \dots, |n/N\rangle, \dots, |(N-1)/N\rangle$$

$$\{|p_m\rangle\} = |0\rangle, |1/N\rangle, \dots, |m/N\rangle, \dots, |(N-1)/N\rangle$$

- Torus topology

$$\psi(q+1) = \langle q+1|\psi\rangle = e^{2\pi i\beta}\psi(q)$$

$$\psi(p+1) = \langle p+1|\psi\rangle = e^{-2\pi i\alpha}\psi(p)$$

- Finite Hilbert space

$$2\pi\hbar = 1/N$$

Semiclassical limit: $\hbar \to 0 \;\; \& \; N \to \infty$

Review quantum maps

- Quantum kicked rotor/ Quantum standard map

$$H(q,p) = \frac{p^2}{2} - \frac{K}{4\pi^2}\cos(2\pi q) \sum_{n=-\infty}^{\infty} \delta(t - n) \quad \Rightarrow \quad U = \exp\left(\frac{iK}{4\pi^2\hbar}\cos(2\pi q)\right) \exp\left(-\frac{ip^2}{2\hbar}\right)$$

- In position representation

$$U_{n'n} = \langle q_{n'}|U|q_n\rangle = \frac{e^{i\pi/4}}{\sqrt{N}} \exp\left(\frac{iKN}{2\pi}\cos\left(\frac{2\pi(n'+\alpha)}{N}\right)\right) \exp\left(\frac{i\pi}{N}(n'-n)^2\right)$$

- Floquet states $U_{n'n}\phi_k(q_n) = e^{i\theta_k}\phi_k(q_{n'})$, → Eigenangles or quasi-energies ($0 \leq \theta_k < 2\pi$)

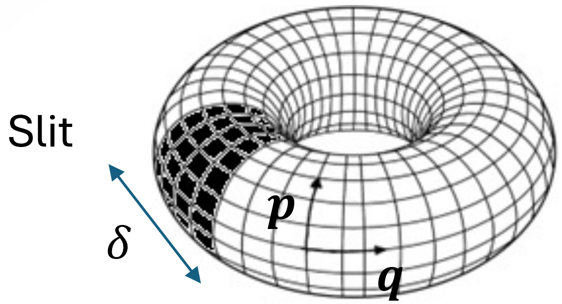
Open quantum maps

- Quantum map with leakage

$$\boxed{\widetilde{U} = U\Pi}$$

Π → Projector on the complement of the hole

U → Closed operator



- Leakeage → strip of width δ parallel to momentum axis

$$\widetilde{U}_{n'n} = \langle q_{n'}|\widetilde{U}\,|q_n\rangle = \sum_{m=0}^{N-1}\langle q_{n'}|\widetilde{U}|p_m\rangle\langle p_m|q_n\rangle = \begin{cases}\widetilde{U}_{n'n} = 0, & q_n \in strip\\ \widetilde{U}_{n'n} = U_{n'n} & q_n \notin strip\end{cases}$$

$\langle p_m|q_n\rangle = 0, q_n \in strip$

$$\widetilde{U}_{n'n} = \begin{pmatrix} U_{n'n} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & U_{n'n}\end{pmatrix}$$

$$\langle \mathrm{q}_{n'}|\Pi|\mathrm{q}_n\rangle = \Pi_{n'n} = \begin{cases}\Pi_{n,n} = 0, q_n \in strip\\ \Pi_{n,n} = 1, q_n \notin strip\end{cases}$$

Eigenvalue Problem

- Closed quantum maps ➔ U is a unitary matrix ➔ Eigenvalues $e^{i\theta_k}$

$$U_{n'n}\phi_k(q_n) = e^{i\theta_k}\phi_k(q_{n'})$$

$e^{i\theta_k}$ → Eigenangles or quasi-energies ($0 \leq \theta_k < 2\pi$)

ϕ_k → Floquet states

- Open quantum maps ➔ $\tilde{U}$ is a non unitary matrix ➔ Eigenvalues $e^{i\theta_k}$ have a complex phase

$$\tilde{U}_{n'n}\phi_k(q_n) = z_k\phi_k(q_{n'}), \qquad z_k = e^{i\theta_k - \Gamma_K/2}$$

$$T_k = 1/\Gamma_k$$

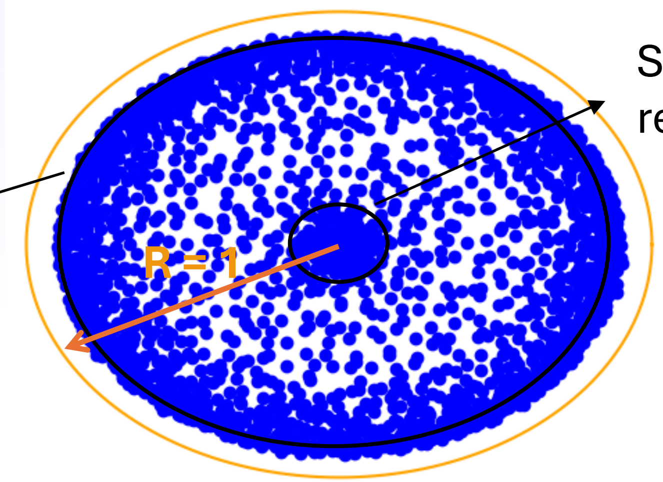
ϕ_k → Resonances

Γ_K → Decay rate

T_k → Dwell time

Resonances

- Eigenvalues are inside the unit disc in the complex plane

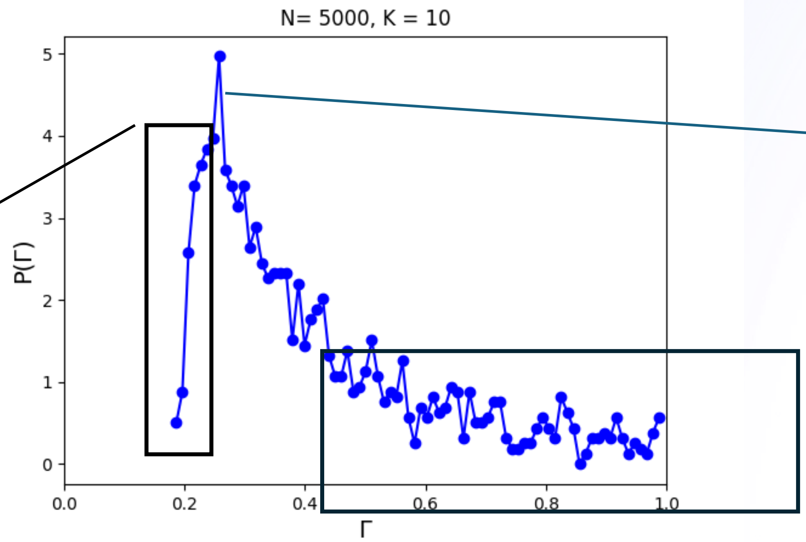


Long-lived resonances

Short-lived resonances

For chaotic regime

- Distribution



Supersharp resonances

$\Gamma_{max} \rightarrow \gamma_c$ (Classical decay rate)

$$\mu_t = e^{-\gamma t}\mu$$

Long tail for $\Gamma > \Gamma_{max}$

Resonances

- Non-unitarity of $\widetilde{U}$ ⟶ Left and Right eigenvectors are different

$$\widetilde{U}\left|\phi_k^R\right\rangle = z_n\left|\phi_k^R\right\rangle \qquad \left\langle\phi_k^L\right|\widetilde{U} = z_n\left\langle\phi_k^L\right|$$

- Trivial right eigenvectors ⟶ $\lambda_k = 0$ ⟶ $\langle q|\phi_k^R\rangle = \phi_k^R\,(q) = (z\;0\;\ldots\;0)^T, \qquad z \in \mathbb{C}$

For chaotic regime

- Right eigenvectors with smaller Γ concentrate on the forward-trapped set.

Left

backward

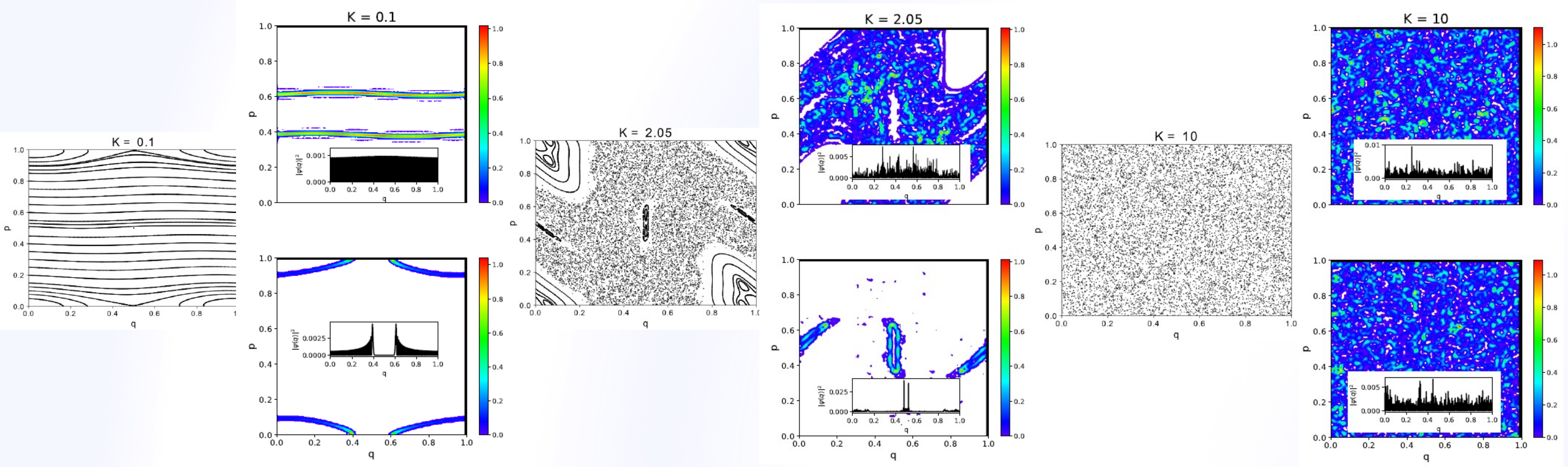
⟶ Trajectories that never hit the hole when iterated foward

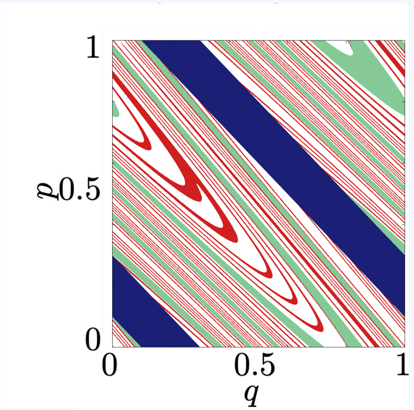
backward

Leaky quantum standard map

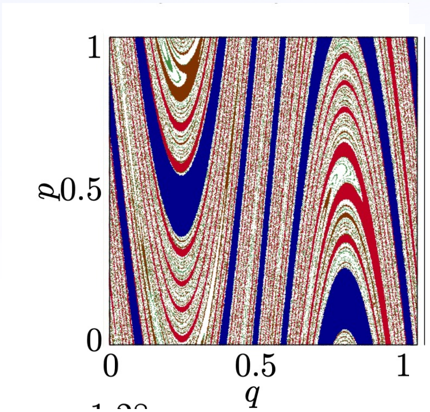
Closed system
Coherent States & Husimi distributions

- $U|q,p\rangle \approx |M(q,p)\rangle, \quad \hbar \to 0$
- $H_k(q,p) = \dfrac{|\langle q,p|\phi_k\rangle|^2}{\langle q,p|q,p\rangle}$

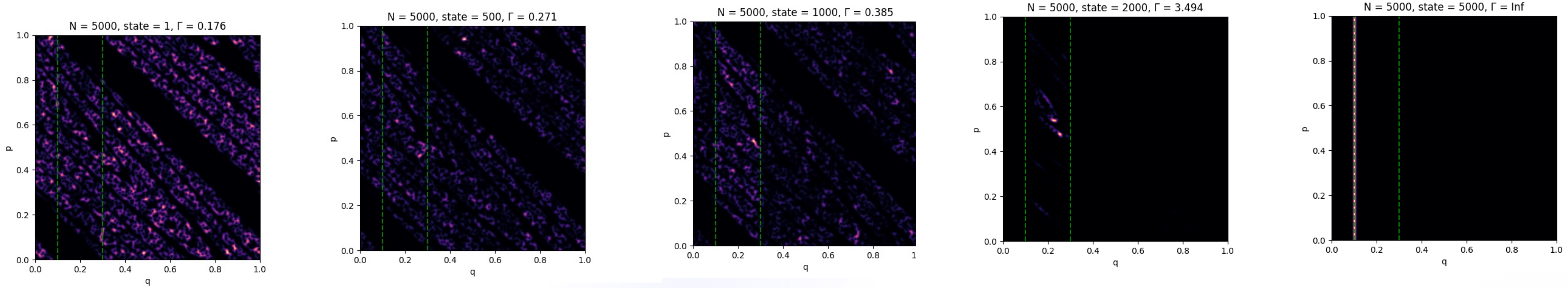




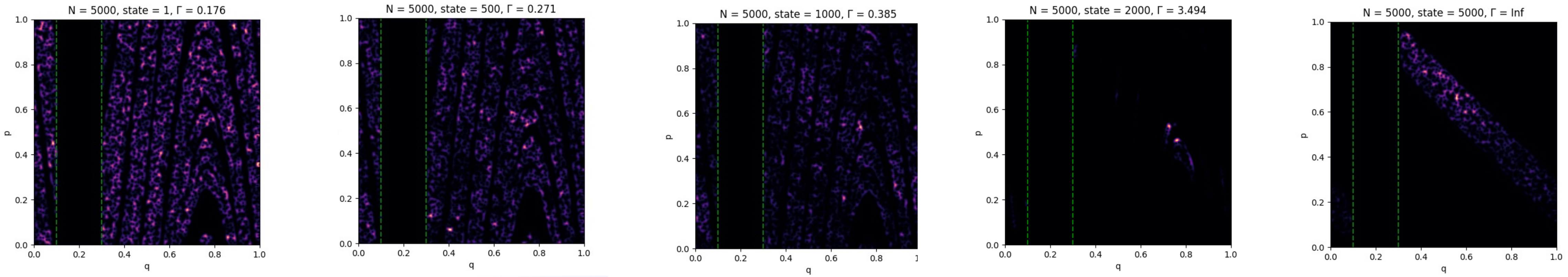
Husimi of the Resonances



Right:



Left:



Schur Decomposition

- Unitary transformation: $\widetilde{U} \rightarrow QTQ^{\dagger}$

$$T = \begin{bmatrix} z_1 & T_{12} & \cdots & T_{1N} \\ 0 & z_2 & \cdots & T_{2N} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & z_N \end{bmatrix}$$

$Q = (v_1\ v_2 \dots v_N)$

↓

$\{|v_k\rangle\} \rightarrow$ orthonormal basis

- Gram-Schmidt process (QR algorithm) of a non-orthogonal set $\{|\phi_k\rangle\}$

I. $|v_1\rangle = |\phi_1\rangle$ ⟶ Depends on $\{|\phi_k\rangle\}$ configuration ⟶ $N!\ \neq$ sets

II. $|v_2\rangle = |\phi_2\rangle - \langle\phi_2|v_1\rangle|v_1\rangle$

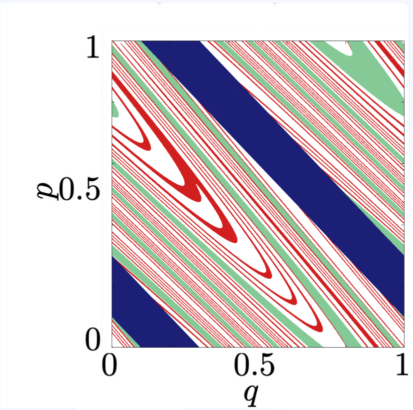
$\vdots$

$$|v_n\rangle = |\phi_n\rangle - \sum_{j=1}^{n-1} \frac{\langle\phi_n|v_j\rangle}{\langle v_j|v_j\rangle}|v_j\rangle$$

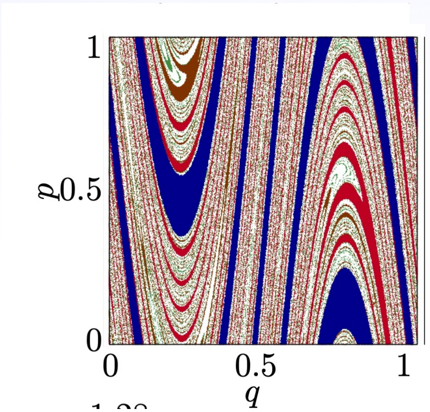
- Select $|z_1| > |z_2| > \cdots > |z_N|$

Long-lived states: $|v_1\rangle$ $|v_2\rangle$

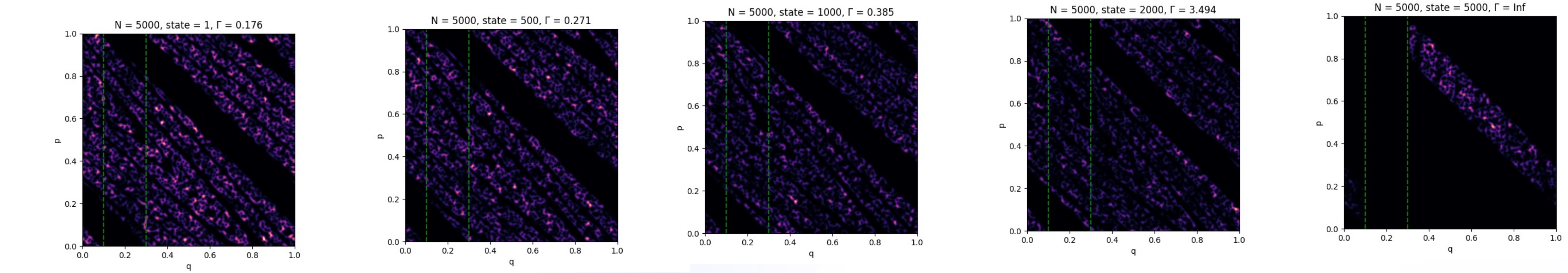
Short-lived states: $|v_N\rangle$



Husimi-Schur distributions



Right:



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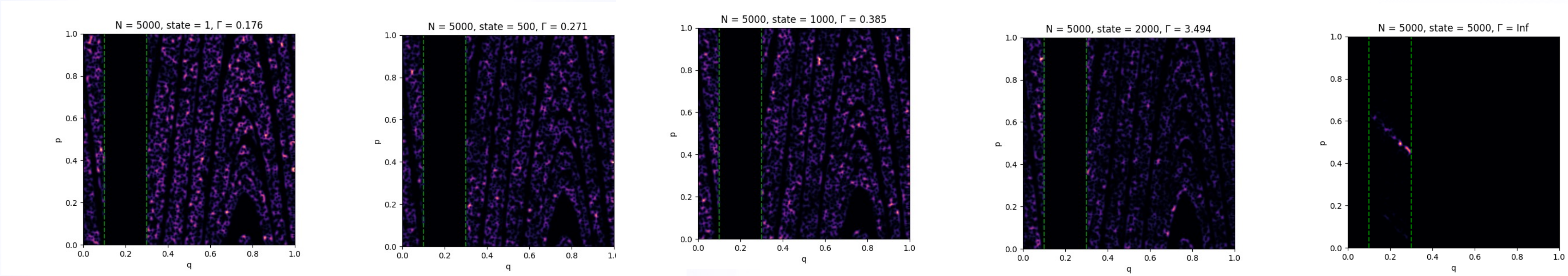
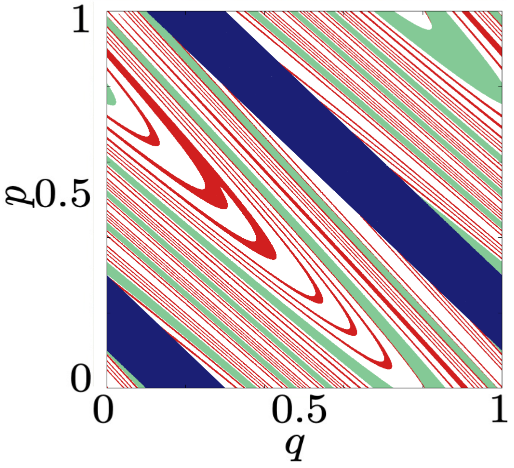
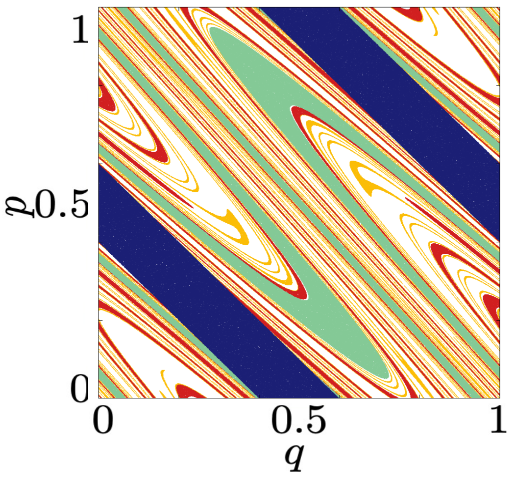


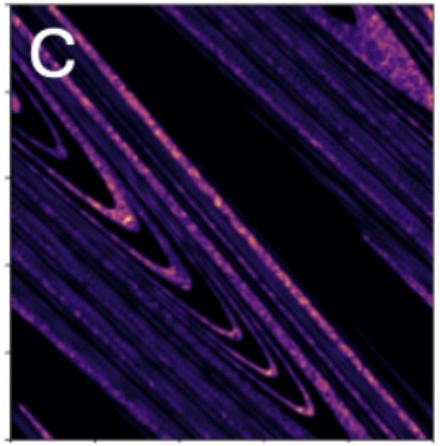
Husimi-Schur distributions

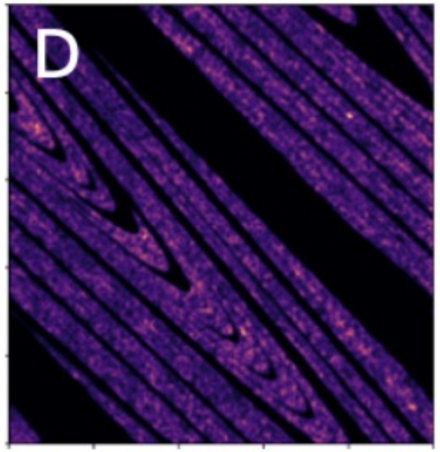
$L_q = 0.2$

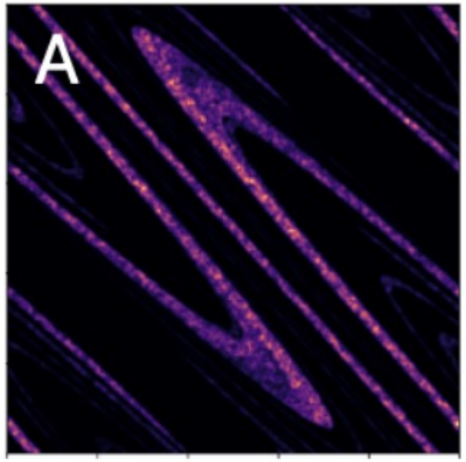


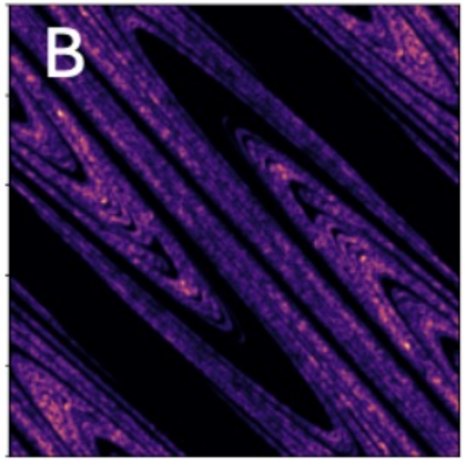
$L_q = 0.5$











Wehrl Entropy

- Quasi-entropy $S_W = -\int H_k(q,p)\log H_k(q,p)\,dqdp$

Quantum maps

$$S_W = -\frac{1}{\mathcal{N}}\sum_{q,p} H_k(q,p)\log H_k(q,p)$$

Normalization constant

$$\text{size}(H_k) = M$$

$$\mathcal{N} = 2\log M$$

- Localization on phase-space:

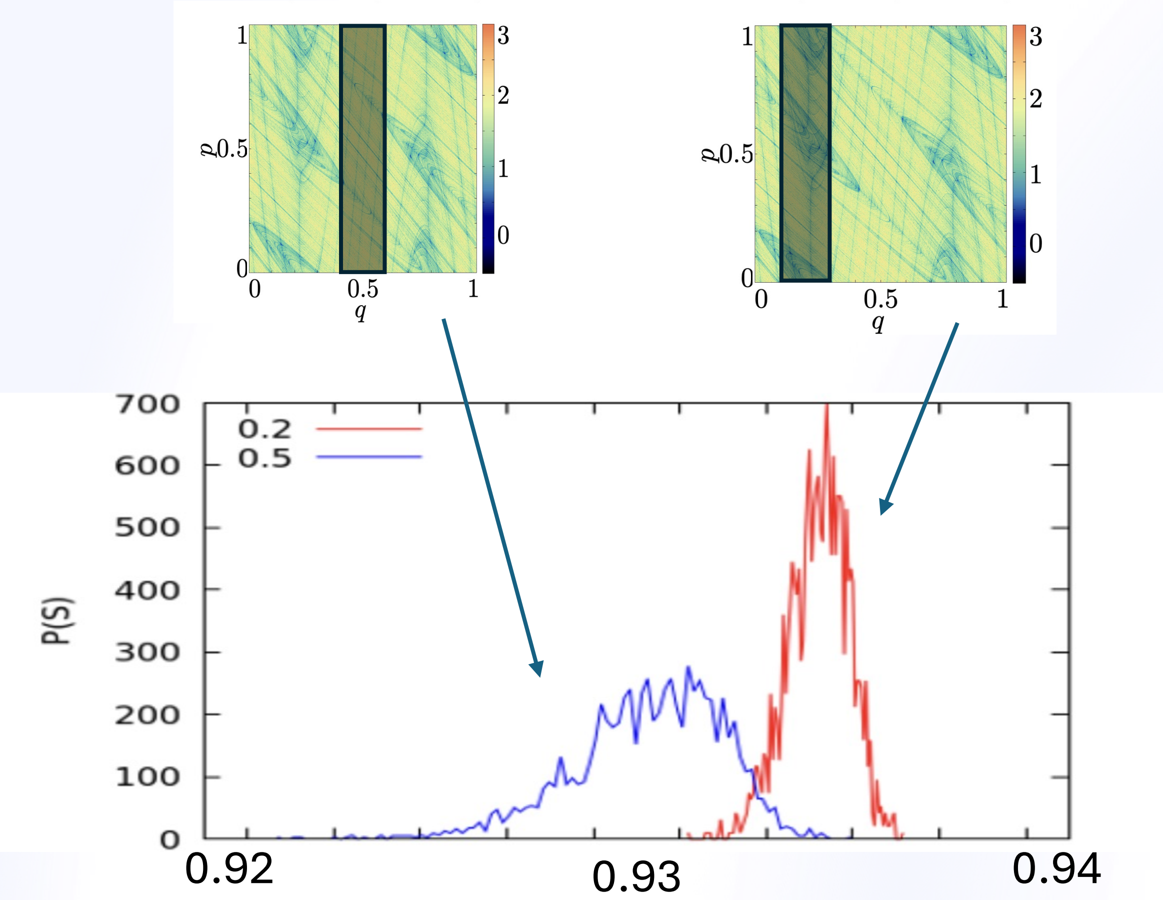
Spread state → Large Entropy

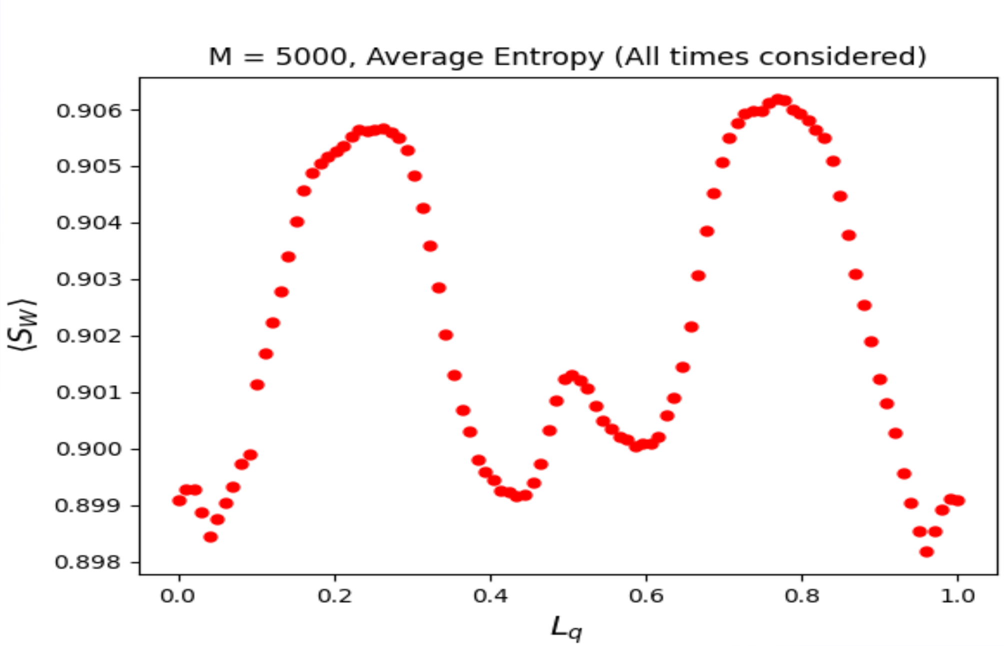
$S_W = 1$ → Fully delocalized state

Localized state → Small Entropy

$S_W = 0$ → Fully localized state

Wehrl Entropy of Husimi - Results





PROBLEMS – Open Quantum map

1. From the Evolution operator in position representation of the quantum standard map, set the matrix in your preferred language code. Now, introduce the slit at $q = [0.2, 0.4]$,which means that for $N = 1000$, the columns $100 - 200$ are zero. Calculate its eigenvalues and check that all fall in the unitary circle. (Suggested parameters $K = 10, N = 1000, \alpha = \beta = 0$).

2. From the previous question, make the distribution of the decay rates Γ_k and check the characteristic distribution. (Sharp growth and long tail).

3. Now, change the parameter $K = 0.3, 2, and\ 5$, see that far from chaos our remarks (Decay rate distribution) are very different due to new classical structures in the system.

4. Finally, calculate the right and left eigenvectors and see that they are not orthogonal.

Thank you