Krylov Complexity and Dynamical Phase Transition in the Lipkin-Meshkov-Glick model

Pedro H. S. Bento

Institute of Physics
Federal University of Goiás

2024

PHYSICAL REVIEW B **109**, 224304 (2024)

Krylov complexity and dynamical phase transition in the quenched Lipkin-Meshkov-Glick model

Pedro H. S. Bento,[1,*] Adolfo del Campo,[2,3] and Lucas C. Céleri[1]
[1] *QPequi Group, Institute of Physics, Federal University of Goiás, 74.690-900, Goiânia, Brazil*
[2] *Department of Physics and Materials Science, University of Luxembourg, L-1511 Luxembourg, Luxembourg*
[3] *Donostia International Physics Center, E-20018 San Sebastián, Spain*

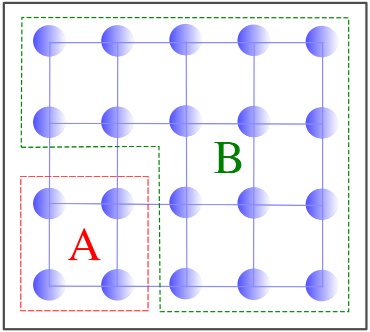
(Received 20 December 2023; revised 29 April 2024; accepted 3 June 2024; published 11 June 2024)

Investigating the time evolution of complexity in quantum systems entails evaluating the spreading of the system's state across a defined basis in its corresponding Hilbert space. Recently, the Krylov basis has been identified as the one that minimizes this spreading. In this study, we develop a numerical exploration of the Krylov complexity in quantum states following a quench in the Lipkin-Meshkov-Glick model. Our results reveal that the long-term averaged Krylov complexity acts as an order parameter for this model. It effectively discriminates between the two dynamic phases induced by the quench, sharing a critical point with the conventional order parameter. Additionally, we examine the inverse participation ratio and the Shannon entropy in both the Krylov basis and the energy basis. A matching dynamic behavior is observed in both bases when the initial state possesses a specific symmetry. This behavior is analytically explained by establishing the equivalence between the Krylov basis and the prequench energy eigenbasis.

DOI: 10.1103/PhysRevB.109.224304

Thermalization in isolated quantum systems

Thermalization in isolated quantum systems



Operator growth

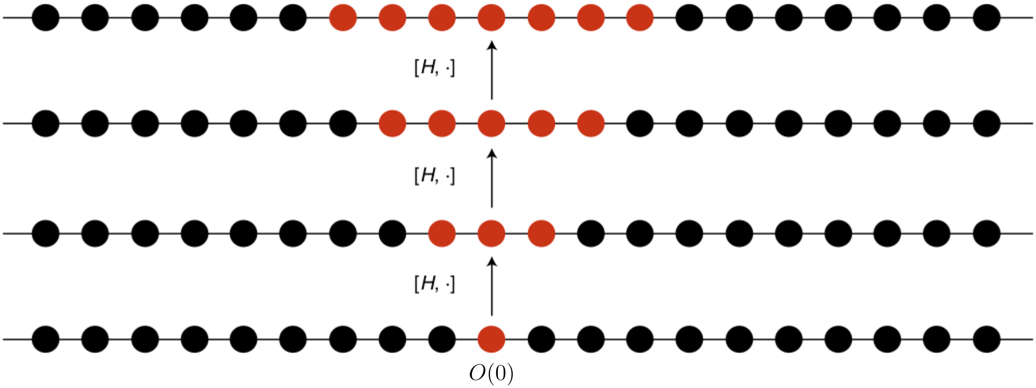
Operator growth

Let us choose an operator $\hat{O}$ which is simple at $t = 0$ and consider its time evolution

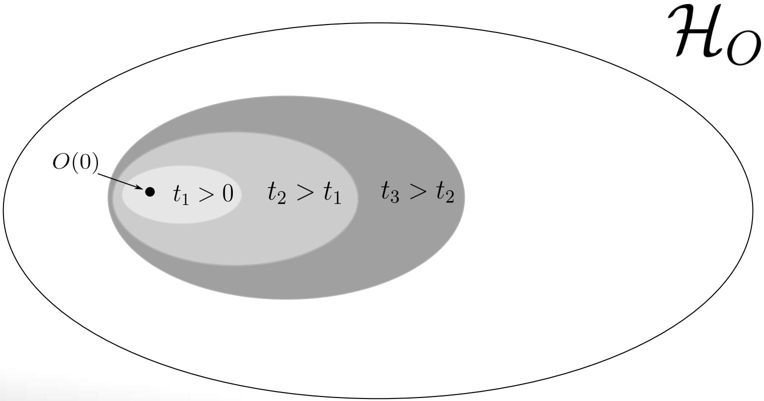
$$\hat{O}(t) = e^{i\hat{H}t}\hat{O}_0 e^{-i\hat{H}t} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\mathcal{L}^n(\hat{O}_0) \tag{1}$$

$$= \hat{O}_0 + it[\hat{H}, \hat{O}_0] + \frac{(it)^2}{2!}[\hat{H}, [\hat{H}, \hat{O}_0]] + \cdots$$

where $\mathcal{L}(\bullet) = [\hat{H}, \bullet]$ is the Liouvillian.


[H, ·]
[H, ·]
[H, ·]
$O(0)$

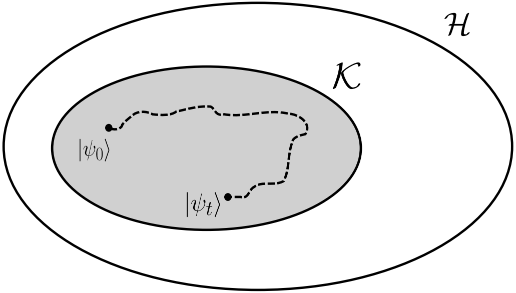
Beyond spatial support growth



Schrödinger picture

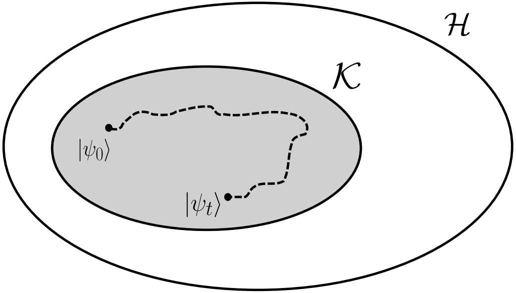
Schrödinger picture

$$|\psi_t\rangle = e^{-i\hat{H}t}|\psi_0\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \hat{H}^n |\psi_0\rangle. \tag{2}$$



Schrödinger picture

$$|\psi_t\rangle = e^{-i\hat{H}t}|\psi_0\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \hat{H}^n |\psi_0\rangle. \quad (2)$$



Problem
How to find the minimal subspace of $\mathcal{H}$ which contains the dynamics of $|\psi_0\rangle$ for any $t > 0$?

The set $\{H^n|\psi_0\rangle\}$ contains all the information only about the portion of $\mathcal{H}$ visited by $|\psi_t\rangle$.

The set $\{H^n|\psi_0\rangle\}$ contains all the information only about the portion of $\mathcal{H}$ visited by $|\psi_t\rangle$.

$$\left\{\hat{H}^n|\psi_0\rangle\right\} \xrightarrow{\text{orthonormalization}} \left\{|K_n\rangle\right\}_{n=0}^{K-1} \tag{3}$$

The set $\{H^n|\psi_0\rangle\}$ contains all the information only about the portion of $\mathcal{H}$ visited by $|\psi_t\rangle$.

$$\left\{\hat{H}^n|\psi_0\rangle\right\} \xrightarrow{\text{orthonormalization}} \left\{|K_n\rangle\right\}_{n=0}^{K-1} \tag{3}$$

If we accomplish this task, we call the set $\left\{|K_n\rangle_{n=0}^{K-1}\right\}$ the **Krylov basis** which spans the so-called **Krylov subspace** $\mathcal{K}_{|\psi_0\rangle}$ for the initial state $|\psi_0\rangle$

Lanczos algorithm - Gram-Schmidt orthogonalization

First step

- $|K_0\rangle = |\psi_0\rangle$
- $|K_1\rangle = \frac{1}{b_1} H|K_0\rangle$, where $b_1 = \sqrt{\langle K_1|K_1\rangle}$

Recursive method:

$$|A_n\rangle = H|K_{n-1}\rangle - a_n|K_{n-1}\rangle - b_{n-1}|K_{n-2}\rangle \tag{4}$$

$$b_n := \sqrt{\langle A_n|A_n\rangle}, \qquad a_n = \langle K_n|\hat{H}|K_n\rangle$$

$$|K_n\rangle = \frac{1}{b_n}|A_n\rangle$$

Output: Lanczos coefficients and the Krylov basis

$$b_n := \sqrt{\langle K_n|K_n\rangle}, \qquad a_n = \langle K_n|\hat{H}|K_n\rangle, \qquad \Big\{|K_n\rangle\Big\}_{n=0}^{K-1}$$

QPEQUI

Interestingly, the hamiltonian in the Krylov basis

$$H \doteq \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_3 & b_3 & \cdots \\ 0 & 0 & b_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5}$$

Krylov subspace method

A method which practically maps any quantum problem into an one dimensional problem.

The tight-binding model

Returning to the Lanczos algorithm, we can write the recurrence in the form

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle. \tag{6}$$

The tight-binding model

Returning to the Lanczos algorithm, we can write the recurrence in the form

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle. \tag{6}$$

Using the Schrödinger equation $i\partial_t|\psi_t\rangle = H|\psi_t\rangle$,

$$i\partial_t\varphi_n(t) = a_n\varphi_n(t) + b_{n+1}\varphi_{n+1}(t) + b_n\varphi_{n-1}(t) \tag{7}$$

where $\varphi(t) := \langle\psi_t|K_n\rangle$ and $\varphi_n(0) = \delta_{n0}$.

The tight-binding model

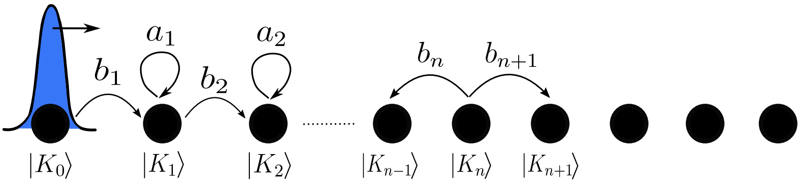
Returning to the Lanczos algorithm, we can write the recurrence in the form

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle. \tag{6}$$

Using the Schrödinger equation $i\partial_t|\psi_t\rangle = H|\psi_t\rangle$,

$$i\partial_t\varphi_n(t) = a_n\varphi_n(t) + b_{n+1}\varphi_{n+1}(t) + b_n\varphi_{n-1}(t) \tag{7}$$

where $\varphi(t) := \langle\psi_t|K_n\rangle$ and $\varphi_n(0) = \delta_{n0}$.



Mapping the problem

The dynamics in the Krylov subspace is equivalent to a hopping particle in a one-dimensional disorded chain.

Mapping the problem

The dynamics in the Krylov subspace is equivalent to a hopping particle in a one-dimensional disorded chain.

As time passes, the evolving state $|\psi_t\rangle$ delocalizes in the Krylov basis. The average position of the *hopping particle* shall reflect this delocalization.

$$C_{\mathcal{K}}(t) = \sum_{n=0}^{K-1} n|\langle\psi_t|K_n\rangle|^2. \tag{8}$$

This quantity has been called **Krylov complexity**.

Summarizing...

- The Krylov subspace is the minimal subspace in which the dynamics of $|\psi_0\rangle$ unfolds.

Summarizing...

- The Krylov subspace is the minimal subspace in which the dynamics of $|\psi_0\rangle$ unfolds.
- In principle, the Krylov subspace may be unique for each initial state $|\psi_0\rangle$ (some states are more "ergodic" than others).

Summarizing...

- The Krylov subspace is the minimal subspace in which the dynamics of $|\psi_0\rangle$ unfolds.
- In principle, the Krylov subspace may be unique for each initial state $|\psi_0\rangle$ (some states are more "ergodic" than others).
- The dynamics in the Krylov subspace can be seen as the delocalization of a single particle wave-packet in a 1D disorded chain with hopping terms given by the Lanczos coefficients.

Summarizing...

- The Krylov subspace is the minimal subspace in which the dynamics of $|\psi_0\rangle$ unfolds.
- In principle, the Krylov subspace may be unique for each initial state $|\psi_0\rangle$ (some states are more "ergodic" than others).
- The dynamics in the Krylov subspace can be seen as the delocalization of a single particle wave-packet in a 1D disorded chain with hopping terms given by the Lanczos coefficients.
- The average position of the hopping particle reflects the delocalization of $|\psi_t\rangle$ in the Krylov subspace and it is called Krylov complexity.

PHYSICAL REVIEW X **9**, 041017 (2019)

A Universal Operator Growth Hypothesis

Daniel E. Parker,[1,*] Xiangyu Cao,[1,†] Alexander Avdoshkin,[1,‡] Thomas Scaffidi,[1,2,§] and Ehud Altman[1,¶]

[1] *Department of Physics, University of California, Berkeley, California 94720, USA*

[2] *Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada*

(Received 17 January 2019; published 23 October 2019)

We present a hypothesis for the universal properties of operators evolving under Hamiltonian dynamics in many-body systems. The hypothesis states that successive Lanczos coefficients in the continued fraction expansion of the Green's functions grow linearly with rate α in generic systems, with an extra logarithmic correction in 1D. The rate α—an experimental observable—governs the exponential growth of operator complexity in a sense we make precise. This exponential growth prevails beyond semiclassical or large-N limits. Moreover, α upper bounds a large class of operator complexity measures, including the out-of-time-order correlator. As a result, we obtain a sharp bound on Lyapunov exponents $\lambda_L \leq 2\alpha$, which complements and improves the known universal low-temperature bound $\lambda_L \leq 2\pi T$. We illustrate our results in paradigmatic examples such as nonintegrable spin chains, the Sachdev-Ye-Kitaev model, and classical models. Finally, we use the hypothesis in conjunction with the recursion method to develop a technique for computing diffusion constants.

DOI: 10.1103/PhysRevX.9.041017

Subject Areas: Condensed Matter Physics, Nonlinear Dynamics, Quantum Physics

The study of Krylov complexity has been applied in

- Several quantum many-body systems (integrable, quasi-integrable and chaotic);
- Distinguishing topological phases of matter;
- Probing equilibrium phase transitions;
- Open quantum systems
- Quantum field theories;
- Cosmology etc;

The study of Krylov complexity has been applied in

- Several quantum many-body systems (integrable, quasi-integrable and chaotic);
- Distinguishing topological phases of matter;
- Probing equilibrium phase transitions;
- Open quantum systems
- Quantum field theories;
- Cosmology etc;

Our motivation

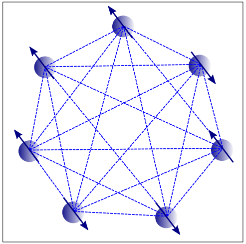
How the Krylov complexity behaves in a cenario of dynamical criticality in isolated systems?

The Lipkin-Meshkov-Glick model

$$\hat{H}(h) = -\frac{J}{2N}\sum_{i<j}^{N} s_i^z s_j^z - h\sum_{i=1}^{N} s^x. \tag{9}$$

In the fully symmetric sector ($j = N/2$), the hamiltonian can be written in terms of collective spin variables $\hat{S}^\alpha = \sum_i s_i^\alpha/2$

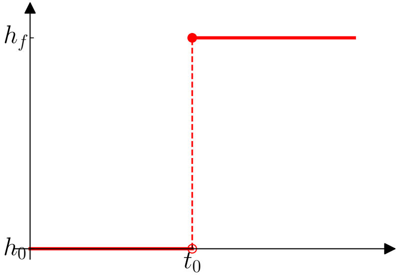
$$\hat{H}(h) = -\frac{J}{N}\hat{S}_z^2 - h\hat{S}_x \tag{10}$$

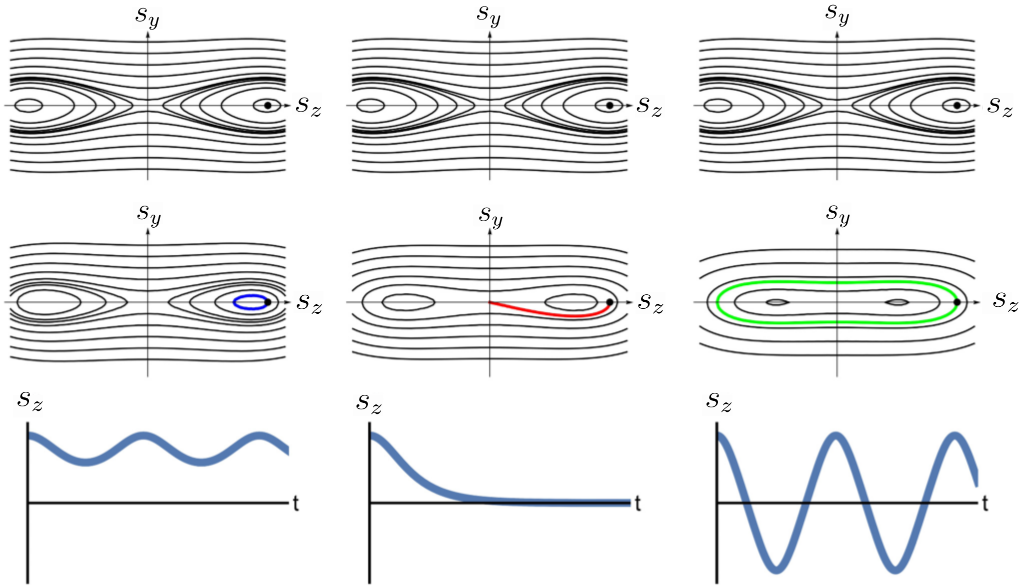


Quench protocol

- $t < t_0$.
  $h = h_0 \longrightarrow \hat{H}(h = h_0) \equiv \hat{H}_0$
  $|\psi(t=0)\rangle \equiv |\psi_0\rangle = |E_0\rangle$, where
  $|E_0\rangle$ is the ground-state of $\hat{H}_0$.

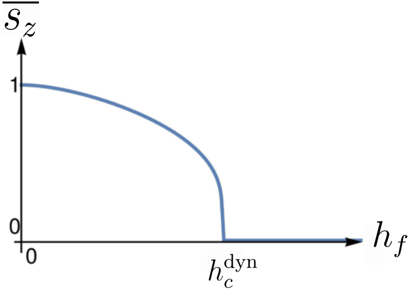
- $t \geq t_0$
  $h = h_f \longrightarrow \hat{H}(h = h_f) \equiv \hat{H}_f$
  $|\psi_t\rangle = e^{-i\hat{H}_f t}|\psi_0\rangle$




S_y
S_z
S_z
t

Dynamical order parameter

$$\overline{s_z} = \lim_{N\to\infty} \frac{1}{N} \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \psi_t | \hat{S}_z | \psi_t \rangle \mathrm{d}t \tag{11}$$

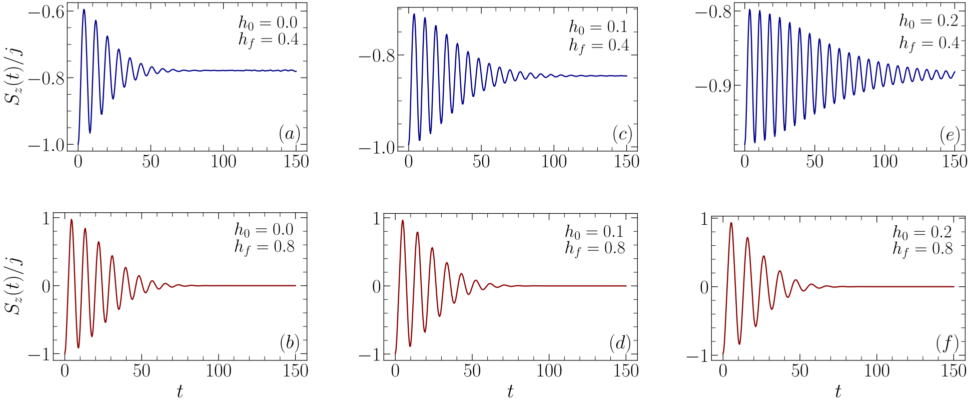


For the LMG model,

$$h_c^{\text{dyn}} = \frac{h_0 + J}{2} \tag{12}$$

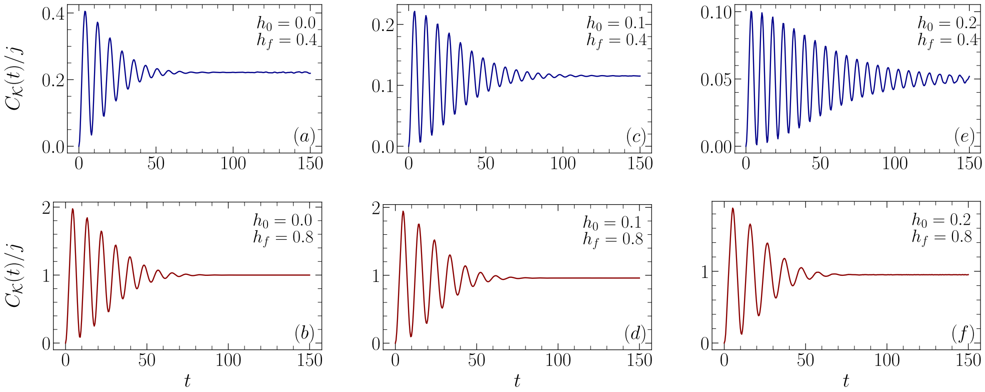
Magnetization in the LMG model

Initial state: South pole of the Bloch sphere (negative magnetization) for $N = 200$.



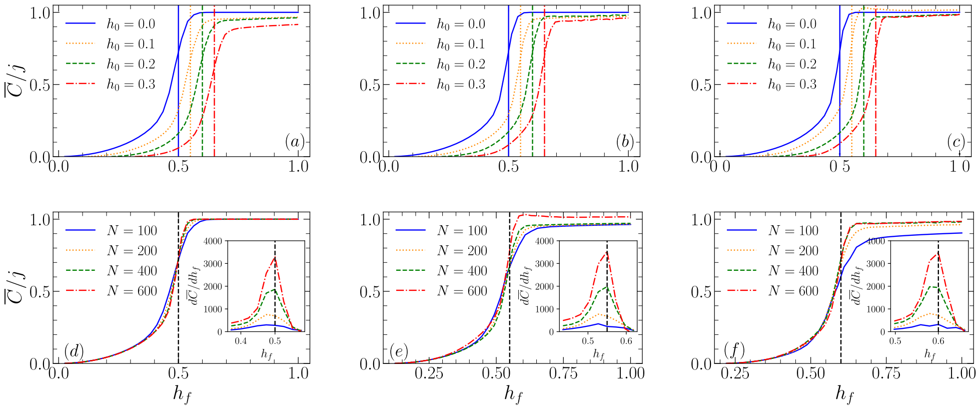
Our results

Krylov complexity in the LMG model



Time-averaged Krylov complexity as an order parameter!

$$\overline{C} = \lim_{T\to\infty} \frac{1}{T}\int_0^T C_{\mathcal{K}}(t)\mathrm{d}t \qquad (13)$$



Once the DPT-I is related to the restoration of symmetry, we can infer that the Krylov basis is also sensitive to the symmetry of the model and, therefore, a deeper relation between the energy basis and Krylov basis should exist.

Inverse Participation Ratio

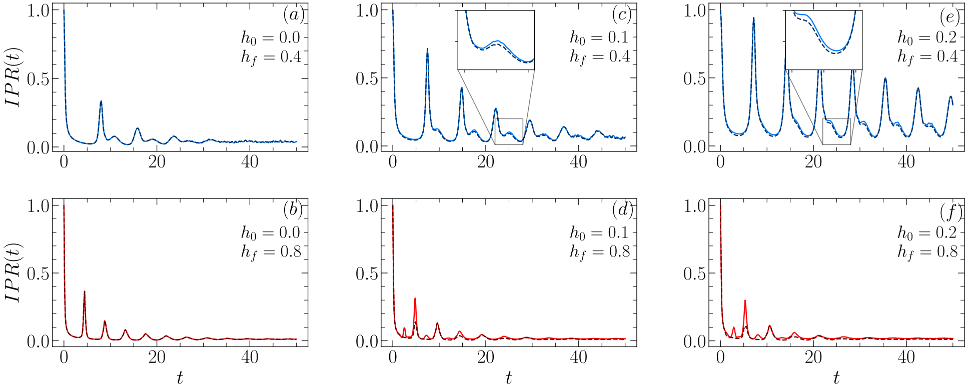
$$\text{IPR}(t) = \sum_k |\langle k|\psi_t\rangle|^4 = \sum_k p_k^2(t). \tag{14}$$

We considered two basis: the Krylov basis $\{|K_n\rangle\}_{n=0}^{K-1}$ and pre-quench energy basis $\{|E_n^0\rangle\}_{n=0}^{N+1}$.

Inverse Participation Ratio

$$\text{IPR}(t) = \sum_k |\langle k|\psi_t\rangle|^4 = \sum_k p_k^2(t). \tag{14}$$

We considered two basis: the Krylov basis $\{|K_n\rangle\}_{n=0}^{K-1}$ and pre-quench energy



Shannon entropy

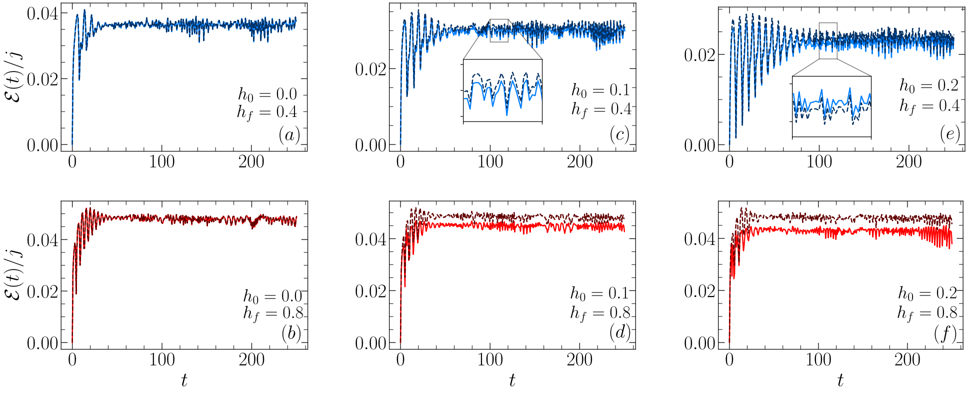
$$\mathcal{E}(t) = -\sum_n p_n(t) \log[p_n(t)] \tag{15}$$

where $p_n(t) = |\langle K_n|\psi(t)\rangle|^2$ for the Krylov basis and $p_n(t) = |\langle E_n^0|\psi(t)\rangle|^2$ for the pre-quench energy basis.

Shannon entropy

$$\mathcal{E}(t) = -\sum_n p_n(t) \log[p_n(t)] \qquad (15)$$

where $p_n(t) = |\langle K_n|\psi(t)\rangle|^2$ for the Krylov basis and $p_n(t) = |\langle E_n^0|\psi(t)\rangle|^2$ for



What is going on here??? Let’s take a closer look at the hamiltonian.

Krylov basis for $h_0 = 0$

Krylov basis for $h_0 = 0$

The pre-quench hamiltonian and its basis are

$$\hat{H}_0 = -\frac{1}{2j}\hat{S}_z^2 \quad \longrightarrow \quad \hat{H}_0|j,m_z\rangle = -\frac{m_z^2}{2j}|j,m_z\rangle, \quad m_z \in \{-j,-j+1,\cdots,j-1,j\}$$

$$\hat{H}_f = -\frac{1}{2j}\hat{S}_z^2 - \frac{h_f}{2}\left(\hat{S}_+ + \hat{S}_-\right)$$

Krylov basis for $h_0 = 0$

The pre-quench hamiltonian and its basis are

$$\hat{H}_0 = -\frac{1}{2j}\hat{S}_z^2 \quad \longrightarrow \quad \hat{H}_0|j, m_z\rangle = -\frac{m_z^2}{2j}|j, m_z\rangle, \quad m_z \in \{-j, -j+1, \cdots, j-1, j\}$$

$$\hat{H}_f = -\frac{1}{2j}\hat{S}_z^2 - \frac{h_f}{2}\left(\hat{S}_+ + \hat{S}_-\right)$$

The post-quench hamiltonian acts on these states yielding

$$\hat{H}_f|j, m_z\rangle = c_0|j, m_z\rangle + c_+|j, m_z+1\rangle + c_-|j, m_z-1\rangle, \tag{16}$$

where

$$c_0 = -m_z^2/2j$$

$$c_\pm = -\frac{h}{2}\sqrt{j(j+1) - m_z(m_z \pm 1)}$$

Krylov basis for $h_0 = 0$

Comparing these last equations

$$\hat{H}_f|j, m_z\rangle = c_0|j, m_z\rangle + c_+|j, m_z + 1\rangle + c_-|j, m_z - 1\rangle \tag{17}$$

with the Lanczos recursive method:

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle \tag{18}$$

Krylov basis for $h_0 = 0$

Comparing these last equations

$$\hat{H}_f|j, m_z\rangle = c_0|j, m_z\rangle + c_+|j, m_z + 1\rangle + c_-|j, m_z - 1\rangle \tag{17}$$

with the Lanczos recursive method:

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle \tag{18}$$

we conclude two final results

Krylov basis for $h_0 = 0$

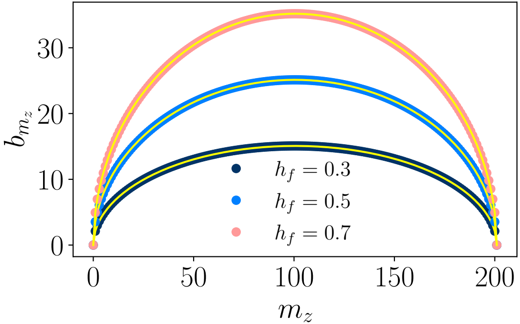
$$|K_i\rangle = \pm|j, m_z\rangle \tag{19}$$

Krylov basis for $h_0 = 0$

$$|K_i\rangle = \pm|j, m_z\rangle \quad (19)$$

and by making the change $m_z \to -j + m_z$

$$b_{m_z} = \frac{h_f}{2}\sqrt{m_z(2j - m_z + 1)}, \qquad m_z \in \{0, 1, \cdots, 2j\} \quad (20)$$



Moreover, note that we can write the expression of the Krylov complexity in the form

$$C_{\mathcal{K}}(t) = \sum_{m_z=0}^{2j} m_z |\langle \psi_t | K_{m_z} \rangle|^2. \tag{21}$$

Considering now the expression of $S_z(t)$,

$$S_z(t) = \langle \psi_t | \hat{S}_z | \psi_t \rangle. \tag{22}$$

and employing the completeness of the angular momentum basis, $\sum_{m_z=0}^{2j} |j, -j + m_z\rangle\langle j, -j + m_z| = \mathbb{1}$, we can readily show that

$$C_{\mathcal{K}}(t) = S_z(t) + j, \tag{23}$$

thus, by taking the long-time average, confirming that

$$\overline{C} = \overline{S_z} + j. \tag{24}$$

QPEQUI

Conclusions and open questions

Conclusions and open questions

- The Krylov subspace method is a powerful tool to understand the dynamics in quantum systems.

Conclusions and open questions

- The Krylov subspace method is a powerful tool to understand the dynamics in quantum systems.
- Krylov complexity safely detects and characterizes dynamical phase transition in the LMG model.

Conclusions and open questions

- The Krylov subspace method is a powerful tool to understand the dynamics in quantum systems.
- Krylov complexity safely detects and characterizes dynamical phase transition in the LMG model.

Open questions

- What is the Krylov basis for $h_0 > 0$?
- How does $\overline{C}$ scales near the dynamical critical point? Is it like $\frac{d\overline{C}}{dh_f} \sim |h - h_f|^\gamma$? What is the value of γ?
- Does $\overline{C}$ functions as an order parameter in other models?
- Deeper connections with ESQPT?

QPEQUI

Thank you for your attention

Contact: pedrosantosbento@gmail.com

www.qpequi.com